Zeros of self-reciprocal polynomials and canonical systems of differential equations

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Self-Reciprocal Polynomials

A polynomial of degree n

$$P(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \quad (c_i \in \mathbb{C},\ c_0 \neq 0).$$

is called a **self-reciprocal** if $x^n P(1/x) = P(x)$,
i.e., $c_k = c_{n-k}$ for every $0 \leq k \leq n$.

Self-Reciprocal Polynomials

- We treat only self-reciprocal polynomials of **even degree**, $2g$, together with **real coefficients**.

  If $P(x)$ is self-reciprocal and of odd degree, then we have

  $$P(x) = (x+1)^r \tilde{P}(x) \quad (r \in \mathbb{Z}_{>0})$$

  for some self-reciprocal polynomial $\tilde{P}(x)$ of even degree.
- We often denote by $P_g(x)$ a self-reciprocal polynomial of degree $2g$ with real coefficients $c_0, c_1, \cdots, c_g$.
- We study conditions of (real) coefficients $c_0, \cdots, c_g$ for $P_g(x)$ having all its zeros on the unit circle S^1.

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Sources of self-reciprocal polynomials

1. The zeta function of a smooth projective curve $C/\mathbb{F}_q$, genus g:

$$Z_C(T) = \frac{Q_C(T)}{(1-T)(1-qT)}.$$

$P_C(x) := Q_C(x/\sqrt{q})$ is a self-reciprocal polynomial of degree $2g$ in $\mathbb{R}[x]$ by the functional equation of $Z_C(T)$.

All zeros of $P_C(x)$ are on the unit circle by a result of A. Weil.

2. The partition function of a ferromagnetic Ising model:
Let $A = (a_{i,j})$ be a $n \times n$ real symmetric matrix with $|a_{i,j}| \leq 1$ for $1 \leq i < j \leq n$. Then

$$P_A(x) = \sum_{k=0}^{n} \Big[\sum_{\substack{I \subset \{1,2,\cdots,n\} \\ |I|=k}} \prod_{i \in I} \prod_{j \notin I} a_{i,j} \Big] x^k$$

is a self-reciprocal polynomial.

All zeros of $P_A(x)$ are on the unit circle by Lee-Yang circle theorem.

3. Discretization of integral formulas of (self-dual) L-functions:

$$\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty f(x)(x^{s-\frac{1}{2}} + x^{-(s-\frac{1}{2})})\frac{dx}{x}$$

$$= \lim_{T\to\infty}\lim_{q\to 1^+}\log q \sum_{k=0}^{\lfloor\frac{\log T}{\log q}\rfloor} f(q^k)(q^{ikz} + q^{-ikz})$$

where $f(x) = \frac{1}{2}\sqrt{x}\frac{d}{dx}\left[x^2\frac{d}{dx}\sum_{n\in\mathbb{Z}}\exp(-\pi n^2x^2)\right]$, $s = \frac{1}{2} - iz$.
The RHS gives a family of self-reciprocal polynomials

$$P_{g,T}(x) = \log q\sum_{k=0}^{g} f(q^k)(x^{g+k} + x^{g-k}), \quad q = T^{\frac{1}{g}}.$$

• Other examples are Alexander polynomials of knots, Duursma zeta polynomials of self-dual codes, etc.

Known General Results

A. Cohn, 1922

All zeros of a self-reciprocal polynomial $P(x) \in \mathbb{R}[x]$ are on S^1 if and only if all the zeros of $P'(x)$ lie inside or on S^1.

Hence, for example, one can test whether all zeros of $P(x)$ are on S^1 by calculating the Mahler measure of $P'(x)$.

Known General Results

A simple condition in terms of coefficients is:

P. Lakatos, 2002

Let $P(x) \in \mathbb{C}[x]$ be a self-reciprocal polynomial of degree $n \geq 2$. Suppose that

$$|c_0| \geq \sum_{k=1}^{n-1} |c_k - c_0|.$$

Then all zeros of $P(x)$ are on the unit circle S^1.

This sufficient condition is generalized by A. Schinzel (2005), Lakatos–L. Losonczi (2009), and D. Y. Kwon (2011).

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W. Chen, 1995; K. Chinen, 2008

Suppose that $P(x) \in \mathbb{R}[x]$ has the form

$$P(x) = (c_0x^n + c_1x^{n-1} + \cdots + c_kx^{n-k}) + (c_kx^k + c_{k-1}x^{k-1} + \cdots + c_0),$$

with $c_0 > c_1 > \cdots > c_k > 0$ $(n \geq k)$.
Then all zeros of $P(x)$ are on the unit circle.

As above, known conditions in terms of coefficients are sufficient conditions. Pattern of coefficients has the form "V" or $|c_0| \approx |c_k|$.

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Result I

- In this talk, we give a necessary and sufficient condition that all zeros of $P(x)$ are on S^1 and simple in terms of coefficients $c_0, \cdots, c_g$ by using the theory of canonical systems of linear differential equations.
- However, the result itself can be stated without the language of canonical systems.

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- However, the result itself can be stated without the language of canonical systems.

Linear System adapted to $P_g(x)$

To state results, we introduce a linear system.
We define matrices $P_k(m_k)$ and Q_k as follows:

$$P_0(m_0) := P_0 := \left[\begin{array}{c|c} 1 & \\ \hline & 1 \end{array}\right], \quad Q_0 := \left[\begin{array}{cc|cc} 1 & 1 & & \\ \hline & & 1 & -1 \end{array}\right],$$

$$P_1(m_1) := \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ \hline 0 & 1 & 0 & -m_1 \end{array}\right], \; Q_1 := \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ \hline & & & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$P_2(m_2) := \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 1 & & & \\ \hline & & & 1 & 0 & 0 \\ & & & 0 & 1 & -1 \\ \hline 0 & 1 & 0 & 0 & -m_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -m_2 \end{array}\right],$$

$$Q_2 := \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & & & & \\ 0 & 1 & 1 & 0 & & & & \\ \hline & & & & 1 & 0 & 0 & -1 \\ & & & & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

For $k \geq 2$, define square matrices $P_k(m_k)$ of size $(2k+2)$ by

$$P_k(m_k) := \begin{bmatrix} V_k^+ & \mathbf{0} \\ \mathbf{0} & V_k^- \\ \mathbf{0}I_k & -m_k \cdot \mathbf{0}I_k \end{bmatrix},$$

where

$$\mathbf{0}I_k := \left[\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{array}\right], \quad m_k \cdot \mathbf{0}I_k := \left[\begin{array}{c|ccc} 0 & m_k & & \\ \vdots & & \ddots & \\ 0 & & & m_k \end{array}\right]$$

and ...

$$V_k^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \left(\tfrac{k+3}{2}\right) \times (k+1),$$

$$V_k^- = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \left(\tfrac{k+1}{2}\right) \times (k+1),$$

if k is **odd**.

$$V_k^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \left(\frac{k+2}{2}\right) \times (k+1),$$

$$V_k^- = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \left(\frac{k+2}{2}\right) \times (k+1),$$

if k is **even**.

Lemma 1

Let $k \geq 1$. We have

$$\det P_k(m_k) = \begin{cases} \pm 2^{j-1} \cdot m_{2j-1}^{j} & \text{if } k = 2j-1, \\ \pm 2^{j} \cdot m_{2j}^{j} & \text{if } k = 2j. \end{cases}$$

In particular, the matrix $P_k(m_k)$ is invertible iff $m_k \neq 0$.

For $k \geq 2$, we define matrices Q_k of size $(2k+2) \times (2k+4)$ by

$$Q_k := \begin{bmatrix} W_k^+ & \mathbf{0} \\ \mathbf{0} & W_k^- \\ \mathbf{0}_{k,k+2} & \mathbf{0}_{k,k+2} \end{bmatrix},$$

$$W_k^+ := \left[\begin{array}{c|c} V_k^+ & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \qquad \begin{cases} \left(\dfrac{k+3}{2}\right) \times (k+2) & \text{if } k \text{ is odd,} \\ \left(\dfrac{k+2}{2}\right) \times (k+2) & \text{if } k \text{ is even,} \end{cases}$$

$$W_k^- := \left[\begin{array}{c|c} V_k^- & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \qquad \begin{cases} \left(\dfrac{k+1}{2}\right) \times (k+2) & \text{if } k \text{ is odd,} \\ \left(\dfrac{k+2}{2}\right) \times (k+2) & \text{if } k \text{ is even.} \end{cases}$$

For a self-reciprocal polynomial
$P_g(x) = c_0 x^{2g} + c_1 x^{2g-1} + \cdots + c_g x^g + \cdots$
and arbitrary fixed $q > 1$,
we define the column vector $v_g(0)$ of length $(4g+2)$ by

$$v_g(0) := \begin{pmatrix} \mathbf{a}_g \\ \mathbf{b}_g \end{pmatrix}, \quad \mathbf{a}_g := \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{g-1} \\ c_g \\ c_{g-1} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix}, \quad \mathbf{b}_g := \begin{pmatrix} c_0 \log(q^g) \\ c_1 \log(q^{g-1}) \\ \vdots \\ c_{g-1} \log q \\ 0 \\ -c_{g-1} \log q \\ \vdots \\ -c_1 \log(q^{g-1}) \\ -c_0 \log(q^g) \end{pmatrix}.$$

We define column vectors $v_g(n)$ of length $(4g+2-2n)$ inductively for $1 \leq n \leq 2g$ by taking $v_g(0)$ as the initial vector :

$$m_{2g-n} := \frac{v_g(n-1)[1] + v_g(n-1)[2g-n+2]}{v_g(n-1)[2g-n+3] - v_g(n-1)[4g-2n+4]},$$

$$v_g(n) := P_{2g-n}(m_{2g-n})^{-1} \cdot Q_{2g-n} \cdot v_g(n-1),$$

where $v[j]$ means j-th component of a column vector v.

- m_{2g-n} and components of $v_g(n)$ are rational functions of coefficients $(c_0, \cdots, c_g)$ and $\log q$ for every $1 \leq n \leq 2g$.
- m_{2g-n} is not identically zero for every $1 \leq n \leq 2g$.

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Statement of Result I

Theorem 1

Let $P_g(x)$ be a self-reciprocal polynomial of degree $2g$ $(g \geq 1)$ with real coefficients $c_0, \cdots, c_g$. Fix $q > 1$ arbitrary and define numbers $m_{2g-n} = m_{2g-n}(c_0, \cdots, c_g; \log q)$ as above.

Then all zeros of $P_g(x)$ are on the unit circle and **simple** if and only if

$$0 < m_{2g-n} < \infty \quad \text{for every } 1 \leq n \leq 2g. \qquad (*)$$

The condition $(*)$ is independent of the choice of $q > 1$.
It is obvious from the following refinement of the above theorem.

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Refinement of Theorem 1

Put $m_{2g} := \dfrac{1}{g \log q}$. Define $R_n = R_n(c_0, \cdots, c_g)$ by

$$R_0 := 1$$

and

$$R_n := \begin{cases} \dfrac{m_{2g-1} m_{2g-3} \cdots m_{2g-2j-1}}{m_{2g} m_{2g-2} \cdots m_{2g-2j}} & \text{if } n = 2j+1, \\ \\ \dfrac{m_{2g-2} m_{2g-4} \cdots m_{2g-2j-2}}{m_{2g-1} m_{2g-3} \cdots m_{2g-2j-1}} & \text{if } n = 2j+2, \end{cases}$$

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We have $R_1(c_0, \cdots, c_g) = 1$ and find that $R_n(c_0, \cdots, c_g)$ depends only on $(c_0, \cdots, c_g)$ for every $0 \leq n \leq 2g$. Moreover,

$$m_{2g-n} = \frac{1}{g \log q} R_{n-1}(c_0, \cdots, c_g) R_n(c_0, \cdots, c_g) \quad (1 \leq \forall n \leq 2g).$$

Theorem 1'

All zeros of $P_g(x)$ are on the unit circle and simple if and only if

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Examples for small g

General algebraic formula of $R_n(c_0, \cdots, c_g)$ is not yet obtained. But for small g, we can calculate them by an algorithmic way:

- $g = 1$, $R_2(c_0, c_1) = \dfrac{2c_0 + c_1}{2c_0 - c_1}$.
- $g = 2$, $R_n = R_n(c_0, c_1, c_2)$ $(2 \leq n \leq 4)$,

$$R_2 = \frac{4c_0 + c_1}{4c_0 - c_1}, \quad R_3 = \frac{8c_0^2 - 2c_1^2 + 4c_0c_2}{8c_0^2 + c_1^2 - 4c_0c_2},$$
$$R_4 = \frac{2c_0 + 2c_1 + c_2}{2c_0 - 2c_1 + c_2}.$$

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$$R_2 = \frac{4c_0 + c_1}{4c_0 - c_1}, \quad R_3 = \frac{8c_0^2 - 2c_1^2 + 4c_0c_2}{8c_0^2 + c_1^2 - 4c_0c_2},$$

$$R_4 = \frac{2c_0 + 2c_1 + c_2}{2c_0 - 2c_1 + c_2}.$$

Examples for small g

General algebraic formula of $R_n(c_0, \cdots, c_g)$ is not yet obtained. But for small g, we can calculate them by an algorithmic way:

- $g = 1$, $R_2(c_0, c_1) = \dfrac{2c_0 + c_1}{2c_0 - c_1}$.
- $g = 2$, $R_n = R_n(c_0, c_1, c_2)$ $(2 \leq n \leq 4)$,

$$R_2 = \frac{4c_0 + c_1}{4c_0 - c_1}, \quad R_3 = \frac{8c_0^2 - 2c_1^2 + 4c_0c_2}{8c_0^2 + c_1^2 - 4c_0c_2},$$
$$R_4 = \frac{2c_0 + 2c_1 + c_2}{2c_0 - 2c_1 + c_2}.$$

- $g = 3$, $R_n = R_n(1, c_1, c_2, c_3)$ $(2 \leq n \leq 6)$,

$$R_2 = \frac{6 + c_1}{6 - c_1}, \quad R_3 = \frac{18 - 3c_1^2 + 6c_2}{18 + 2c_1^2 - 6c_2},$$

$$R_4 = \frac{36 + 6c_1 - c_1^2 + 4c_1^3 - 4c_2^2 + 18c_3 - 14c_1c_2 + c_1^2c_2 + 3c_1c_3}{36 - 6c_1 - c_1^2 - 4c_1^3 - 4c_2^2 - 18c_3 + 14c_1c_2 + c_1^2c_2 + 3c_1c_3},$$

$$\begin{aligned} R_5 = (&108 - 21c_1^2 - 12c_1^4 + 108c_2 - 12c_2^2 - 12c_2^3 - 27c_3^2 \\ &+ 42c_1^2c_2 + 3c_1^2c_2^2 - 54c_1c_3 - 6c_1^3c_3 + 30c_1c_2c_3) \\ /(&108 + 9c_1^2 + 8c_1^4 - 108c_2 + 36c_2^2 - 4c_2^3 - 27c_3^2 \\ &- 42c_1^2c_2 + c_1^2c_2^2 + 54c_1c_3 - 4c_1^3c_3 + 18c_1c_2c_3), \end{aligned}$$

$$R_6 = \frac{2 + 2c_1 + 2c_2 + c_3}{2 - 2c_1 + 2c_2 - c_3}.$$

An interpretation of Theorem 1

The positivity of numbers m_{2g-n} or R_n $(1 \le n \le 2g)$ can be interpreted from the viewpoint of the theory of canonical systems of linear differential equations.

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Now we review the theory of canonical systems of linear differential equations in order to explain the above things.

Definition of Canonical Systems

Let $H(a)$ be a 2×2 matrix-valued function defined almost everywhere on an interval $I = [1, a_0)$ $(1 < a_0 \leq \infty)$.

A family of linear differential equations on I of the form

$$-a\frac{d}{da}\begin{bmatrix} A(a,z) \\ B(a,z) \end{bmatrix} = z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H(a) \begin{bmatrix} A(a,z) \\ B(a,z) \end{bmatrix}, \quad \lim_{a \to a_0^-} \begin{bmatrix} A(a,z) \\ B(a,z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

parametrized by the complex parameter $z \in \mathbb{C}$ is called a (two-dimensional) **canonical system** if

- $H(a) = {}^tH(a)$ and $H(a) \geq 0$ for almost every $a \in I$,
- $H(a) \not\equiv 0$ on any open subset $J \subset I$ with $|J| > 0$,
- $H(a)$ is locally integrable on I.

For a canonical system, $H(a)$ is called its **Hamiltonian**.

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Relation with Theorem 1

By using the previous m_{2g-n} and $v_g(n)$ attached to $P_g(x)$, we can construct a 2×2 matrix-valued function $H_q(a)$ and a pair of functions $(A_q(a,z), B_q(a,z))$ satisfying a system of linear differential equations on $[1, q^g)$ such that $H_q(a)$ is expected to be its Hamiltonian.

Construction of the Hamiltonian

Define the function $m_q(a)$ on $[1, q^g)$ by

$$m_q(a) = m_{2g-n} \quad \text{if} \quad q^{\frac{n-1}{2}} \le a < q^{\frac{n}{2}}$$

and define the 2×2 matrix-valued function $H_q(a)$ by

$$H_q(a) = \begin{bmatrix} m_q(a)^{-1} & 0 \\ 0 & m_q(a) \end{bmatrix}.$$

By Theorem 1, $H_q(a)$ satisfies conditions of a Hamiltonian if and only if all zeros of $P_g(x)$ are on the unit circle and simple.

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Construction of the solution

In addition, define $A_q(a,z)$ and $B_q(a,z)$ for $(a,z) \in [1, q^g) \times \mathbb{C}$ by

$$\begin{bmatrix} A_q(a,z) \\ B_q(a,z) \end{bmatrix} := \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \times$$
$$\begin{bmatrix} c_g(a,z) & \cdots & c_{g-(2g-n)}(a,z) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s_g(a,z) & \cdots & s_{g-(2g-n)}(a,z) \end{bmatrix} v_g(n)$$

if $q^{\frac{n-1}{2}} \le a < q^{\frac{n}{2}}$ $(1 \le n \le 2g)$, where

$$c_k(a,z) := 2\cos(z\log(q^k/a)),$$
$$s_k(a,z) := 2i\sin(z\log(q^k/a)).$$

Construction of the boundary value

For a self-reciprocal polynomial $P_g(x)$ and $q > 1$, we define

$$A_q(z) := q^{-giz} P_g(q^{iz}) = \sum_{k=0}^{g-1} c_k \left(q^{(g-k)iz} + q^{-(g-k)iz} \right) + c_g.$$

Clearly, all zeros of $P_g(x)$ are on the unit circle and simple if and only if all zeros of $A_q(z)$ are real and simple.

Further, we define

$$B_q(z) := -\frac{d}{dz} A(z), \qquad E_q(z) := A_q(z) - iB_q(z).$$

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Statement of Result II

Theorem 2

$A_q(a,z)$ and $B_q(a,z)$ are continuous functions on $[1, q^g)$ w.r.t. a, differentiable on $(q^{(n-1)/2}, q^{n/2})$ for $1 \le \forall n \le 2g$, and satisfy

$$-a\frac{d}{da}\begin{bmatrix} A_q(a,z) \\ B_q(a,z) \end{bmatrix} = z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H_q(a) \begin{bmatrix} A_q(a,z) \\ B_q(a,z) \end{bmatrix},$$

$$\begin{bmatrix} A_q(1,z) \\ B_q(1,z) \end{bmatrix} = \begin{bmatrix} A_q(z) \\ B_q(z) \end{bmatrix}, \quad \lim_{a \to q^g} \begin{bmatrix} A(a,z) \\ B(a,z) \end{bmatrix} = E_q(0) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $z \in \mathbb{C}$. Moreover, this is a canonical system if and only if all zeros of $P_g(x)$ are on the unit circle and simple.

Results of L. de Branges

de Branges, I, 1959–1962

Every canonical system has a unique solution $(A(a,z), B(a,z))$ such that $E(a,z) := A(a,z) - iB(a,z)$ is entire w.r.t z, satisfies

$$\text{(HB)} \qquad |F(z)| > |F^{\sharp}(z)| \quad \text{for} \quad \Im(z) > 0 \quad (F^{\sharp}(z) = \overline{F(\bar{z})}),$$

has no real zeros, and $E(a,0) = 1$ for every (regular) $a \in [1, a_0)$.

Condition (HB) implies that all zeros of $A(a,z)$ and $B(a,z)$ are real and interlace.

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Results of L. de Branges

de Branges, II, 1959–1962

Let $E(z)$ be an entire function satisfying (HB), having no real zeros, and $E(0) = 1$. Then there exists a canonical system and its solution $(A(a,z), B(a,z))$ satisfying

$$\begin{bmatrix} A(1,z) \\ B(1,z) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(E(z) + E^{\sharp}(z)) \\ \frac{i}{2}(E(z) - E^{\sharp}(z)) \end{bmatrix}.$$

That is, canonical systems coincide with entire functions satisfying (HB) and having no real zeros.

Lemma 2

$E_q(z) = A_q(z) - iA_q'(z)$ satisfies (HB) and has no real zeros if and only if all zeros of $A_q(z)$ are real and simple.

Therefore, if all zeros of $P_g(x)$ are on S^1 and simple, then there exists a canonical system s.t. its solution $(A(a,z), B(a,z))$ satisfies

$$\begin{bmatrix} A(1,z) \\ B(1,z) \end{bmatrix} = \begin{bmatrix} A_q(z) \\ B_q(z) \end{bmatrix}.$$

Theorem 2 assert that this canonical system and the solution are constructed concretely by using numbers m_{2g-n} and vectors $v_g(n)$.

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Chebyshev transform and Algebraic formula

There exists real numbers $\lambda_1, \cdots, \lambda_g$ such that

$$P_g(x) = \sum_{k=0}^{g-1} c_k(x^{2g-k} + x^k) + c_g x^g$$
$$= c_0\, x^g \prod_{j=1}^{g} (x + x^{-1} - 2\lambda_j).$$

($P_g(x) \mapsto c_0 \prod_{j=1}^{g} (y - 2\lambda_j)$ is called the Chebyshev transform.)

In terms of $\lambda_1, \cdots, \lambda_g$, we have

- $g = 1$, $R_2(c_0, c_1) = \dfrac{1-\lambda_1}{1+\lambda_1}$.
- $g = 2$, $R_n = R_n(c_0, c_1, c_2)$ $(2 \leq n \leq 4)$,

$$R_2 = \frac{(1-\lambda_1)+(1-\lambda_2)}{(1+\lambda_1)+(1+\lambda_2)},$$

$$R_3 = \frac{(1-\lambda_1^2)+(1-\lambda_2^2)}{(\lambda_1-\lambda_2)^2},$$

$$R_4 = \frac{(1-\lambda_1)(1-\lambda_2)}{(1+\lambda_1)(1+\lambda_2)}.$$

- $g = 3$, $R_n = R_n(c_0, c_1, c_2, c_3)$ $(2 \leq n \leq 6)$,

$$R_2 = \frac{(1-\lambda_1)+(1-\lambda_2)+(1-\lambda_3)}{(1+\lambda_1)+(1+\lambda_2)+(1+\lambda_3)},$$

$$R_3 = \frac{(1-\lambda_1^2)+(1-\lambda_2^2)+(1-\lambda_3^2)}{(\lambda_1-\lambda_2)^2+(\lambda_1-\lambda_3)^2+(\lambda_2-\lambda_3)^2},$$

$$R_4 = \frac{\sum_{1\leq i<j\leq 3}(1-\lambda_i)(1-\lambda_j)(\lambda_i-\lambda_j)^2}{\sum_{1\leq i<j\leq 3}(1+\lambda_i)(1+\lambda_j)(\lambda_i-\lambda_j)^2},$$

$$R_5 = \frac{\sum_{1\leq i<j\leq 3}(1-\lambda_i^2)(1-\lambda_j^2)(\lambda_i-\lambda_j)^2}{\prod_{1\leq i<j\leq 3}(\lambda_i-\lambda_j)^2},$$

$$R_6 = \frac{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}{(1+\lambda_1)(1+\lambda_2)(1+\lambda_3)}.$$

As mentioned before, general (algebraic) formula of $R_n(c_0, \cdots, c_g)$ or $R_n(\lambda_1, \cdots, \lambda_g)$ had not yet been obtained.

We can deal with the case that all zeros of $P_g(x)$ are on the unit circle but $P_g(x)$ may have multiple zeros, if we define m_{2g-n} by taking the column vector $v_g(0) = \begin{pmatrix} \mathbf{a}_{g,\omega} \\ \mathbf{a}_{g,\omega} \end{pmatrix}$ of length $(4g+2)$ as the initial vector, where

$$\mathbf{a}_{g,\omega} = {}^t(c_0 q^{g\omega} \ c_1 q^{(g-1)\omega} \ \cdots \ c_{g-1} q^{\omega} \ c_g \ c_{g-1} q^{-\omega} \ \cdots \ c_0 q^{-g\omega})$$

for $\omega > 0$. In this way, we obtain a family of systems parametrized by $\omega > 0$ which corresponds to the family of functions

$$\frac{1}{2}\Big(A_q(z+i\omega) + A_q(z-i\omega)\Big) \quad (\omega > 0)$$

as well as $A_q(z)$.